1. INTRODUCTION

The measurements of directional-hemispherical transmittance and reflectance are widely used in present-day identification procedures for obtaining information on the main radiative properties of semitransparent disperse media. In many cases, it is assumed that the relation between the measured values of $T_{d,h}$ and $R_{d,h}$ and characteristics of medium absorption and scattering can be described by the radiation transfer theory. According to this theory, radiation transfer depends on the following spectral parameters of the medium: index of refraction $n$, absorption coefficient $\alpha$, scattering coefficient $\sigma$, and single-scattering (phase) function $\Phi(\mu_0)$, where $\mu_0 = \cos \theta_0$, $\theta_0$ being the angle of scattering. For concise writing, we will omit the subscript $\lambda$ in designations of spectral values. Using the known value of the sample thickness $d$, one can consider the single-scattering albedo $\omega = \alpha / \beta$ and optical thickness $\tau_0 = \beta d$ ($\beta = \alpha + \sigma$ is the extinction coefficient) instead of $\alpha$ and $\sigma$. It is clear that it is difficult to determine all the values $n, \omega, \tau_0$, and $\Phi(\mu_0)$ on the basis of directional-hemispherical measurements. Fortunately, the index of refraction is usually known from independent measurements, and the effect of the scattering function on hemispherical transmittance and reflectance can be described by the so-called transport approximation:

\[ \Phi(\mu_0) = 1 - \bar{\mu} + 2\bar{\mu}\delta(1 - \mu_0), \]

where $\bar{\mu}$ is the asymmetry factor of scattering defined as $\bar{\mu} = \frac{1}{2}\int_{-1}^{1} \Phi(\mu_0) \mu_0 d\mu_0$ and $\delta$ is the Dirac function used to model the forward-scattering peak typical for large particles. The transport approximation, which has been used first in the neutron transport theory, allows reducing the radiation transfer problem to the form similar to that for isotropic scattering but with transport scattering coefficient $\sigma_t$ instead of $\sigma$. This procedure is known also as the isotropic scaling. As a result, we have only two parameters to be determined: $\omega = \sigma_t / \beta_t$ and $\tau_0 = \beta_t d$.

The transport approximation is widely employed in radiation heat transfer problems. It is known that this approach is sufficiently accurate for calculating the thermal radiation in many applications. At the same time, the applicability of the transport approximation or isotropic scaling for problems with collimated incident radiation is not evident. It has been recently shown by Tagne and Baillie that error of the scaling may be considerable. In the present paper, we continue this analysis and consider new results for refracting media. In addition, we suggest a modification of a two-flux method applicable to the case of collimated irradiation of a refracting sample. After separation of collimated and diffuse components of the radiation field, the angular dependence of diffuse radiation intensity is approximated in this method by a step function referenced to the critical angle of reflection at the interface.

2. RADIATION TRANSFER PROBLEM

Consider the problem of radiation transfer in a plane-parallel layer of an absorbing, refracting, and scattering medium. We will limit our consideration to the one-
dimensional azimuthally symmetric problem when one surface of the layer is uniformly illuminated along the normal direction by randomly polarized radiation. In the case of a homogeneous isotropic medium, the radiation transfer equation (RTE) and the associated boundary conditions can be written as follows:\(^3\):

\[
\frac{\partial \bar{I}}{\partial \tau} + \bar{I} = \frac{\omega_{\tau}}{4\pi} \int_{-1}^{1} \bar{I}(\tau', \mu') \int_{0}^{2\pi} \Phi(\mu_{0}) d\psi' d\mu',
\]

\(\bar{I}(0, \mu) = R\bar{I}(0, -\mu) + (1 - R) \delta(1 - \mu),\)

where \(\mu = \cos \vartheta\), the angle \(\vartheta\) being measured from the normal directed into the medium; \(\bar{I} = \frac{I_{i}}{(n^2\mu_{s})}, I_{s}\) being the incident spectral radiation intensity; \(\tau = \beta z; \mu_{0} = \mu_{s} \mu + (1 - \mu^2)^{1/2}(1 - \mu^2)^{1/2} \cos \varphi'\); and \(R(\mu)\) is the Fresnel reflection coefficient.\(^4\) In the transport approximation, problem (2)–(3) is reduced to the following one:

\[
\frac{\partial \bar{J}}{\partial \tau_{tr}} + \bar{J} = \frac{\omega_{\tau}}{2} \int_{-1}^{1} \bar{I} d\mu,
\]

\(\bar{J}(0, \mu) = R\bar{J}(0, -\mu) + (1 - R) \delta(1 - \mu),\)

\(\bar{J}(\tau_{tr}, \mu) = R\bar{J}(\tau_{tr}, -\mu), \quad \mu > 0.\)

Following the usual technique,\(^1,3,5\) we present the radiation intensity \(I\) as a sum of the diffuse component \(\bar{J}\) and the term that corresponds to the transmitted and reflected directional external radiation:

\[
\bar{I} = \bar{J} + \frac{1 - R_{1}}{1 - R_{1}C_{tr}} \left\{ \exp(-\tau_{tr}) \delta(1 - \mu) + C_{tr} \exp(\tau_{tr}) \delta(1 + \mu) \right\},
\]

where \(C_{tr} = R_{1} \exp(-2\tau_{tr})\) and \(R_{1} = R(1)\). The index of absorption of the host medium is assumed to be small in comparison with the index of refraction. In this case, one can use the approximate formula \(R_{1} = (n - 1)^2/(n + 1)^2\).

The mathematical problem statement for the diffuse component of radiation intensity is as follows:

\[
\frac{\partial \bar{J}}{\partial \tau_{tr}} + \bar{J} = \frac{\omega_{\tau}}{2} \left[ \int_{-1}^{1} \bar{I} d\mu + \frac{1 - R_{1}}{1 - R_{1}C_{tr}} \left( \exp(-\tau_{tr}) \right) \right.
\]

\[
+ \left. C_{tr} \exp(\tau_{tr}) \right\},
\]

\(\bar{J}(0, \mu) = R(\mu)\bar{J}(0, -\mu), \quad \bar{J}(\tau_{tr}, -\mu) = R(\mu)\bar{J}(\tau_{tr}, -\mu), \quad \mu > 0.\)

The directional-hemispherical reflectance and transmittance can be also expressed through the diffuse component of the radiation intensity:

\[
R_{d-h} = R_{d-h}^0 + \int_{0}^{1} \left[ 1 - R(\mu) \right] \bar{J}(0, -\mu) d\mu,
\]

\[
T_{d-h} = T_{d-h}^0 + \int_{0}^{1} \left[ 1 - R(\mu) \right] \bar{J}(\tau_{tr}, -\mu) d\mu,
\]

where the first terms are given by the well-known equations\(^3\):

\[
R_{d-h}^0 = \frac{R_{1} + (1 - R_{1})^2 C_{tr}}{1 - R_{1}C_{tr}}, \quad T_{d-h}^0 = \frac{(1 - R_{1})^2}{1 - R_{1}C_{tr}} \exp(-\tau_{tr}).
\]

It is important that the source term on the right-hand side of Eq. (7) does not depend on angular coordinate \(\mu\). It permits the further simplification of the problem. Note that separation of collimated radiation in the case of an arbitrary scattering function [problem (2)–(3)] leads to the following relations instead of Eqs. (6) and (7):

\[
\bar{I} = \bar{J} + \frac{1 - R_{1}}{1 - R_{1}C_{tr}} \left\{ \exp(-\tau_{tr}) \delta(1 - \mu) + C \exp(\tau_{tr}) \delta(1 + \mu) \right\},
\]

\[
\frac{\partial \bar{J}}{\partial \tau_{tr}} + \bar{J} = \frac{\omega_{\tau}}{2} \left[ \int_{-1}^{1} \bar{J} d\mu \right] f(\mu, \mu') d\mu' + \frac{1 - R_{1}}{1 - R_{1}C_{tr}} C_{tr} \Phi(\mu) \exp(-\tau_{tr})
\]

\[
+ \Phi(-\mu) C \exp(\tau_{tr}) \right\},
\]

where

\[
f(\mu, \mu') = \frac{1}{2\pi} \int_{0}^{2\pi} \Phi(\mu_{0}) d\psi', \quad C = R_{1} \exp(- 2\tau_{tr}).
\]

The source function on the right-hand side of Eq. (11) depends on \(\mu\), and it makes the problem much more complicated than that in the transport approximation.

3. ESTIMATION OF THE TRANSPORT APPROXIMATION ERROR

In the case of spherical particles or long fibers with a known size distribution, one can use the Mie theory to predict the scattering function of the medium.\(^5\) In many other cases, the scattering function of a disperse medium is unknown, and it is difficult to find the transport approximation error. We will assume that this error can be evaluated by a comparison of calculations for two model scattering functions: the transport one [Eq. (1)] and the Heneyy–Greenstein function:

\[
\Phi(\mu_{0}) = (1 - \mu^2)(1 + \mu^2 - 2 \mu_{0})^{3/2}.
\]

It is clear that the difference between these two approximations is negligible at a small asymmetry factor and increases with \(\mu\) mainly because of quite different backward scattering. The latter is expected to be important for calculating the reflectance.

Note that some real scattering functions cannot be well approximated by relation (14), and one needs a more ad-
equate approximation for exact radiation transfer calculations.12 In these cases, as has been recently shown by Tagné and Bailleux,10 the transport approximation may be even better than the Henyey–Greenstein one.

Numerical solution of the RTE for both transport and Henyey–Greenstein scattering functions can be obtained by use of the discrete ordinates method3 (DOM). It is known that Fresnel’s reflection may cause the so-called ray effect associated with insufficiently fine angular discretization of the radiation intensity field.3,13 Ray effects may be mitigated by refining the angular discretization or by using the modifications of the DOM. In our case, an adequate account for the angular dependence can be reached by use of the composite DOM (CDOM) when the integral over direction is split into integrals over three subintervals by the critical angle, and each subinterval uses a set of quadrature points.14 Note that a similar numerical method has been used recently by Muresan et al.15 for a radiative-conductive heat transfer problem in a nonscattering medium.

The effect of the scattering function on directional-hemispherical transmittance and reflectance is illustrated by CDOM calculations presented in Figs. 1 and 2. The large value of \( \bar{\mu} \) is chosen to evaluate the upper limit of the difference between two approximations of the scattering function. One can see that results depend considerably on index of refraction. For a nonrefracting medium \((n = 1)\), the effect of the scattering function on transmittance is very small in comparison with the effect on reflectance. In the case of \( n = 1.4 \), the value of \( T_{d-h} \) is more sensitive to the scattering function, whereas the results on \( R_{d-h} \) obtained in the transport approximation and in the Henyey–Greenstein approximation are closer to each other.

Assuming the scattering function is equal to the Henyey–Greenstein one, consider the value of relative errors of the transport approximation \( \epsilon_T = T_{d-h}^{tr}/T_{d-h}^{HG} - 1 \) and \( \epsilon_R = R_{d-h}^{tr}/R_{d-h}^{HG} - 1 \), where the superscripts denote the calculations for the corresponding scattering functions. The dependences of \( \epsilon_T, \epsilon_R \) on albedo and optical thickness shown in Fig. 3 confirm the results for isotropic scaling accuracy obtained by Tagné and Bailleux10 in the case of nonrefracting media. Note that the transport approximation can be better than the Henyey–Greenstein one for real scattering functions with considerable backscattering.10 The data for sensitivity of the directional-hemispherical transmittance and reflectance to the scattering function should be taken into account in the procedure of identification of the radiative properties. This problem will be discussed in Section 5 of the paper.

4. MODIFICATION OF THE TWO-FLUX APPROXIMATION FOR REFRACTING MEDIA

The numerical procedure based on the CDOM code is general and can be applied to rather complicated problems.
But in the case of the transport scattering function, the angular dependencies of the diffuse radiation component are expected to be rather simple. For this reason, we consider an alternative approach, which is a modification of the well-known two-flux approximation. Taking into account the effect of total internal reflection on both interfaces of the medium layer, we suggest the following approximation \(^{16,17}\):

\[
\tilde{J}(\tau_r, \mu) = \begin{cases} 
\phi_0(\tau_r), & -1 \leq \mu < -\mu_c \\
\psi_0(\tau_r), & -\mu_c < \mu < \mu_c, \quad \mu_c = (1 - 1/h^2)^{1/2} \\
\phi_0(\tau_r), & \mu_c < \mu \leq 1
\end{cases}
\]

(15)

Note that the case \(\mu_c = 0\) corresponds to the usual two-flux model. The intermediate angle interval \(-\mu_c < \mu < \mu_c\) gives no contribution to the radiation flux, and the words “two-flux” are applicable to the modified approximation, too. It is clear that relation (15) is just the same as in the CDOM of the zero-order quadrature.

Integrating Eq. (7) separately over the intervals \(-1 < \mu < -\mu_c\), \(-\mu_c < \mu < \mu_c\), and \(\mu_c < \mu < 1\), after simple transformations, one can obtain the following boundary-value problem for function \(g_0 = \varphi_0 + \phi_0\):

\[
-\tilde{g}_0'' + \kappa^2 \tilde{g}_0 = \kappa^2 \chi [\exp(-\tau_r) + C_{tr} \exp(\tau_r)],
\]

(16)

\[
(1 + \mu_c)\tilde{g}_0'(0) = 2 \gamma\tilde{g}_0(0), \quad (1 + \mu_c)\tilde{g}_0'(\tau_r) = -2 \gamma \tilde{g}_0'(\tau_r),
\]

where

\[
\kappa^2 = \frac{4}{(1 + \mu_c)^2} \frac{1 - \omega_{tr}}{1 + \omega_{tr}}, \quad \gamma = \frac{1 - R_1}{1 + R_1}, \quad \chi = \frac{\omega_{tr}}{1 - \omega_{tr}} \frac{1 - R_1}{1 - R_{1} C_{tr}}.
\]

(18)

Approximate equations for the reflectance and transmittance of the medium are written as

\[
R_{d-h} = R_{d-h}^0 + \gamma(1 - \mu_c^2)g_0^0/2,
\]

(19)

\[
T_{d-h} = T_{d-h}^0 + \gamma(1 - \mu_c^2)g_0^0/2.
\]

The boundary-value problem [Eqs. (16) and (17)] can be solved analytically. Note that the particular solutions of inhomogeneous equation (16) are as follows:

\[
g_0^* = \frac{\tau_r}{2} \chi [\exp(-\tau_r) + C_{tr} \exp(\tau_r)] \quad \text{when} \quad \kappa = 1,
\]

(20)

\[
g_0^* = \frac{\kappa^2}{\kappa^2 - 1} \chi [\exp(-\tau_r) + C_{tr} \exp(\tau_r)] \quad \text{when} \quad \kappa \neq 1.
\]

(21)

The resulting expressions for \(R_{d-h}\) and \(T_{d-h}\) are different for \(\kappa = 1\) and \(\kappa \neq 1\). In the first case, we have

\[
R_{d-h} = R_{d-h}^0 + D_1 B, \quad T_{d-h} = T_{d-h}^0 + D_1 \left[ A + \phi_{tr} (1 + R_1) E_{tr} \right],
\]

(22)

\[
D_1 = \chi (1 - \mu_c^2)^2/2,
\]

\[
A = \frac{k_1 (\omega s + c) E_{tr} + k_2}{(1 + \omega s + 2 \omega c)}, \quad B = \frac{k_1 E_{tr} + k_2 (\omega s + c)}{(1 + \omega s + 2 \omega c)},
\]

(23)
In the second case, Eqs. (23) is the same, but Eqs. (22) and (24) should be replaced by the following ones:

\[
R_{d-h} = R_{d-h}^0 + D(1 + B/\kappa + C_{tr}),
\]

\[
T_{d-h} = T_{d-h}^0 + D[A/\kappa + (1 + R_1)E_{tr}],
\]

\[
D = D(\kappa^2/(\kappa^2 - 1)),
\]

\[
k_1 = (1 - 2\gamma) - (1 + 2\gamma)R_1, \quad k_2 = (1 - 2\gamma)C_{tr} - (1 + 2\gamma).
\]

In Eqs. (22)–(25), the following designations are used:

\[
\varphi = 2\gamma/\kappa, \quad \gamma = \gamma/(1 + \mu_c),
\]

\[
E_{tr} = \exp(-\frac{\varphi}{\tau_{tr}}), \quad s = \sinh(\kappa\tau_{tr}), \quad c = \cosh(\kappa\tau_{tr}).
\]

A comparison between the analytical solution [Eqs. (22)–(27)] and the numerical results obtained by use of the high-order CDOM for the transport scattering function is given in Figs. 4 and 5. One can see that the modified two-flux approximation is rather accurate for both refracting and nonrefracting media, especially in the case of small and moderate optical thicknesses.

**5. COMMENTS ON THE IDENTIFICATION PROCEDURE**

Consider the problem of identification of both \(\tau_{tr}\) (or \(\beta_{tr}\)) and \(\omega_{tr}\) using the directional-hemispherical measurements. The best way to avoid large identification errors is to find two combinations of the measured values \(T_{d-h}, R_{d-h}\) that are much more sensitive to one of two parameters, \(\tau_{tr}\) and \(\omega_{tr}\). The effect of the unknown scattering function on the identification results should also be taken into account. We will limit our consideration to the important cases of optically thin and optically thick samples.

### A. Optically Thin Sample

Analysis of the derived analytical solution for the case of \(\tau_{tr} < 1\) shows that one can use the difference \(T_{d-h} - R_{d-h}\) to identify optical thickness \(\tau_{tr}\) and the sum \(T_{d-h} + R_{d-h}\) for subsequent determination of transport albedo \(\omega_{tr}\). The possibility of separately identifying the transport optical thickness using the difference \(T_{d-h} - R_{d-h}\) is illustrated in Fig. 6. One can see that the effect of the albedo is very small in the range of \(\tau_{tr} < 1\). It means that \(T_{d-h} - R_{d-h} \approx T_{d-h}^0 - R_{d-h}^0\) and the dependences shown in Fig. 6 are well approximated by the following relation:
\[
\tau_{tr} = \ln \frac{(1 - R_1)^2}{R_1 + (T_{d-h} - R_{d-h})}.
\]  

(28)

One should remember that Eq. (28) is based on the modified two-flux approximation. Figure 7 illustrates that the error of simple relation (28) is not too large even in the case of the Henyey–Greenstein scattering function with high asymmetry of scattering.

After determining \(\tau_{tr}^0\), one can identify the value of the albedo by using strong dependences of \(\tau_{tr}^0\) on the sum of transmittance and reflectance at fixed \(\tau_{tr}^0\). The numerical results shown in Fig. 8 confirm that the sum \(T_{d-h} + R_{d-h}\) is insensitive to the scattering function in the range of \(\tau_{tr}^0 < 1\) and can be calculated on the basis of the modified two-flux approximation. One can see that the error of transport albedo identification may be large in the case of a refracting medium of low albedo [Fig. 8(b)], but one can be sure of rather good identification of \(\omega_{tr}\) for highly scattering media (in the range of \(\omega_{tr} > 0.7\)).

As an example of a possible application of approximate relation (28) for \(\tau_{tr}^0(T_{d-h} - R_{d-h})\) and the analytical solution for the dependences \(\omega_{tr}(T_{d-h} + R_{d-h})\), one can remember the identification of near-infrared radiative properties of optically thin fused-quartz samples containing bubbles.

**B. Optically Thick Sample**

In the opposite case of optically thick samples of absorbing and scattering material, one cannot measure small transmittance, but the data for reflectance contain information about the single-scattering albedo and the scattering function of the medium. It is shown in Fig. 2(b) that the direction-hemispherical reflectance of refracting material with index of refraction about \(n = 1.4\) is insensitive to the scattering function and one can determine the transport albedo \(\omega_{tr}\) from the data for \(R_{d-h}\). It is more difficult to study a nonrefracting medium with considerable absorption \(\omega_{tr} = 0.9\) because of the increased role of the scattering function [see Fig. 1(b)]. In the case of a nonabsorbing medium, the value of \(R_{d-h}\) for optically thick samples is very close to 100%, and one cannot obtain any information from the measurements of direction-hemispherical reflectance.

In some applied problems, one does not need even two parameters (\(\beta_{tr}\) and \(\omega_{tr}\)) for calculating the disperse system radiation. The known example is thermal radiation of optically thick scattering coatings. One can remember advanced highly scattering thermal insulations containing hollow microspheres, which are used not only in cryogenic space engineering. The polymer coatings containing hollow glass microspheres are considered an efficient way of decreasing the radiative heat losses from buildings. As has been shown recently by Dombrovsky, the low values of integral hemispherical emissivity of such coatings are completely determined by the transport albedo of the composite medium in the middle-infrared spectral range.

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**Fig. 6.** Dependence of the transport optical thickness on the difference of direction-hemispherical transmittance and reflectance calculated by use of the modified two-flux approximation: (a) \(n = 1\), (b) \(n = 1.4\), \(\omega_{tr} = 0\); 2, 0.25; 3, 0.5; 4, 0.75; 5, 1.

**Fig. 7.** Dependence of the transport optical thickness on the difference of direction-hemispherical transmittance and reflectance. Comparison of the analytical solution for nonscattering medium (1) with the numerical data for the transport (2, 3) and Henyey–Greenstein (4, 5, \(\bar{\mu} = 0.8\)) scattering functions: (a) \(n = 1\), (b) \(n = 1.4\); 2, 4, \(\omega_{tr} = 0.5\); 3, 5, \(\omega_{tr} = 1\).
In the limit of an optically thick layer, the modified two-flux approximation gives the following expression for the directional-hemispherical reflectance:

\[ R_{d-h} = R_1 + (1 - R_1) \frac{\omega_{tr}}{2n^2} \frac{\kappa^2}{1 - \omega_{tr}(1 + \kappa)(2\gamma + \kappa)}. \]  

(29)

The dependences \( R_{d-h}(\omega_{tr}) \) calculated by formula (29) are shown in Fig. 9. One can see that \( R_{d-h} \) is weakly sensitive to the value of refraction index in the range of \( 1.4 < n < 1.6 \) typical for polymers in the middle infrared. A comparison with the exact numerical solution of the RTE for the transport and Henyey–Greenstein scattering functions (see Fig. 9) illustrate low sensitivity of \( R_{d-h} \) to the scattering function and rather high accuracy of approximation (29). At the same time, the dependence \( R_{d-h}(\omega_{tr}) \) is rather strong for highly scattering media (\( \omega_{tr} > 0.8 \)). It means that the measurements of directional-hemispherical reflectance and simple relation (29) can be used to identify the infrared radiative properties of advanced polymer coatings containing hollow microspheres.

6. CONCLUSIONS

A modified two-flux approximation taking into account the Fresnel reflection is suggested for calculating the directional-hemispherical transmittance and reflectance of a refracting, absorbing, and scattering medium. A comparison of the corresponding analytical solution with the exact numerical calculations for the model transport scattering function showed that the error of the modified two-flux approximation is not greater than 5% in the most important range of the problem parameters. The wide-range calculations by use of the composite DOM and the Henyey–Greenstein scattering function permits estimation of the conditions when directional-hemispherical characteristics are insensitive to the scattering function and the transport approximation is applicable.

Possible applications of the derived analytical solution to identification problems are discussed. For small and moderate optical thicknesses, the optimal combination of directional-hemispherical parameters is recommended. The corresponding analytical expression for transport optical thickness is suggested. In the optically thick limit, the approximate analytical relation for the directional-hemispherical reflectance is derived. It is expected that this relation will be useful in the experiment studying the infrared radiative properties of advanced highly scattering polymer coatings containing hollow microspheres. As a result, this work presents a simple method to identify analytically the radiative properties of absorbing and scattering media from hemispherical measurements in two important limiting cases of optically thick and optically thin samples.

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